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ON THE RELATIONSHIP BETWEEN ALGEBRA AND ANALYSIS

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Dedicated to Professor Saunders MacLane

According to a popular definition, algebra is the study of sets X equipped with operations $X^n \to X$ for various values of n, together with equations which hold identically among these operations. The values of n are assumed to be finite. In [16] Mac Lane restricts himself to theories containing the rational operations, essentially addition and multiplication, and this is a restriction which there is some point in making. The algebraic theories most relevant to analysis are those which are extensions of the theory of rings.

It is widely believed that it is the finiteness of combining power of algebraic operations which distinguishes algebra from analysis. For analysis always appears to involve infinite processes.

But drawing the distinction in this way is not convincing. If the values of n above are also allowed to be infinite, then we get the very natural extension of the concept of an algebraic theory [13] known as a varietal theory [15]. Among varietal theories, the strictly algebraic form an interesting subclass, but no technical difficulties or changes of method arise in working in the more general context. Thus the sway of algebra can easily be extended to theories with infinitary operations. (If the combining power of the operations is not bounded by a regular cardinal $\aleph$, free algebras may not exist, but the category of sets $< \aleph$ is a topos in any case.) And there are plenty of other infinite processes in algebra, for example, completions constructed by inverse limit. On the other hand, infinite processes in analysis don't necessarily imply infinite operations; for example, taking the limit of a convergent sequence is probably a one-variable operation relative to some basic structure which subsumes the notion of sequence.

It is probably more accurate to say that infinite processes affect the equations which hold in analysis, not the operations. If we are to prove equality of two constructed analytic quantities, such as two areas A_1 and A_2, then we usually start by making finite approximations to A_1 and A_2 and proving approximate equality

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$\mathrm{approx}(A_1) = \mathrm{approx}(A_2)$. But the law of continuity by which approximate equality is extended to exact equality $A_1 = A_2$ appeals to infinite process.

1. A combinatorial approach to analysis

1.1. Combinatorics and algebra

It may be better to try other ways of understanding the relationship between algebra and analysis. I will use the above definition of algebra, without any special commitment about the combining powers of operations or (as is common nowadays) what category the object referred to as "X" belongs to.

What seems important is that there is a subtheory of algebra in which no uncertainties of the infinite are present and which does shed light on the processes of analysis and the relationship of algebra to that subject. This subtheory I shall call *combinatorics*. By a combinatorial (algebraic) theory I mean a theory in which the free models generated by finite sets are finite. In other words, a combinatorial theory is one which is completely meaningful within the universe of finite sets (the topos of sets $< \aleph_0$, the first infinite regular cardinal).

The relevance of combinatorial theories is that these theories allow us to carry out the operations on finite approximations which are traditionally used to build the processes of differentiation and integration in analysis. Combinatorial theories do not help us to settle the question of equality in analysis. That is a deeper question than can be settled by mathematics alone. It involves our perceptions of the real world with which analysis is supposed to be concerned, and how the real world is to be modeled within mathematics.

Examples of combinatorial theories are the theory of G-sets where G is a finite group or monoid, and the theory of graphs. In both of these subjects counting arguments predominate. The theory of rings, on the other hand, is not combinatorial. The free model generated by a finite set $\{X_1, \ldots, X_n\}$ is the polynomial ring $\mathbb{Z}[X_1, \ldots, X_n]$, which is infinite. Although the operations involve but a finite number of arguments, the fact that operations can be indefinitely composed leads to infinitely many values.

The main combinatorial theory of which we assume knowledge is that of simplicial sets [7, 17, 20]. Simplicial sets are algebraic over graded sets. If $G = (G_p)_{p \geq 0}$ is a graded set, the free simplicial set generated by G is

$$F(G) = \coprod_{p \geq 0} G_p \times \Delta(p)$$

where $\Delta(p)$ denotes the standard p-dimensional (simplicial) simplex $0 \leq 1 \leq 2 \leq \cdots \leq p$. If G is finite (G_p finite for all p, empty for all but a finite number of p), then $F(G)$ is also finite (in the sense of simplicial sets, having only a finite number of nondegenerate simplices). Note that models themselves of a combinatorial theory do not have to be finite, but a theory is combinatorial $\Leftrightarrow$ every model is the union of its finite submodels. Every simplicial set is the union of its finite subcomplexes.

A minor point may bother us. A simplex x generates an infinite number of degenerate simplices by repeated application of the degeneracy operators $x \to s_j x$. But it is easily proved from standard identities that any simplicial relation between degenerate simplices is a consequence of a relation between their nondegenerate cores.

In the simplicial category, $\Delta(p)$ replaces the natural number object. Since simplicial models of space are built up from the $\Delta(p)$, we receive the impression that the natural number object is absorbed into the structure of space [2].

1.2. Combinatorial host continua for analysis

Analysis has two main aspects:

(1) the theory of differentiation and integration, and

(2) the theory of functions.

Volterra, in an address at the International Congress of Mathematicians (Paris, 1900) called the eighteenth century the era of differential and integral calculus, and the nineteenth the era of the theory of functions [22]. (Perhaps the twentieth is the era of penetration of algebraic methods, such as the theory of linear spaces.)

It seems illogical to put (1) before (2), since in modern analysis the operations of differentiation and integration are carried out on functions. But the study of history clearly shows that the founders of the calculus performed these operations directly on a concept of variable geometrical or physical quantity, not on functional representations of quantity in the modern sense [5]. Number-valued functions result from choice of a unit of measurement, a step which was generally avoided in the early geometrical calculus. The process of measurement converts variable quantity into variable number, that is, into real-valued functions $X \to R$, where X is a state space and R is some continuum of "real numbers" or pure quantity. If x is any kind of quantity, its value in R is the ratio x/u where u is the unit of measurement. R has a unit element u/u, and in fact must be a ring, since it must have an operational structure representing the addition and multiplication of quantities in the world.

It is now customary to express all concepts of analysis in terms of R-valued functions. A numerical continuum R which in this way plays a universal role for the concepts of analysis I shall call a *host continuum* for the analysis.

All forms of analysis developed since the time of Euler have depended on host continua. The host continua employed have been varied, for they reflect different opinions analysts have about reality. We list nine current hosts which occur in various forms of analysis. The first seven will be called *familiar* hosts, and the ones most used in this paper will be (1), (6) and (7).

(1) *The classical continuum.* The host is the complete ordered field of classical real numbers. It is constructed by completion of the field of rational numbers. The concept of limit is considered of primary importance.

(2) *The non-standard continuum.* The host is still the classical reals, but the structure is the enlargement $R \to {}^*R$. (Thus ${}^*R$ is a structure, not another continuum).

Many mathematicians believe that "real" should imply "effective." Thus we also have:

(3) *Recursive continua.* (See [9], for example.)

(4) *The intuitionistic continuum.* (The work of L.E.J. Brouwer.)

(5) *The constructivist continuum.* E. Bishop has claimed "numerical content" as a merit of his continuum [3]. This accords well with the increasing use of numerical methods in mathematics. But why not be even more realistic?

(6) *The continuum of infinite decimals.* The structure of this continuum must be a kind of universal branching process. Setting $\varepsilon = 10^{-n}$, $\delta = 10^{-m}$ gives a representation of the classical continuum and its structure within the continuum of infinite decimals, but gives no insight into the structure of the latter.

We may go even further in pursuit of realism.

(7) *Continua of finite decimals.* These continua consist of numbers representable in finite machines and calculable with *a priori* bounded resources. Such *finite continua* will be used a great deal in the sequel, because they are models of the "real numbers" which can be dealt with within combinatorics. We shall denote one of these continua by $C(k)$. Its numerical content will consist of finite decimal *displays* with a fixed number, k, of decimal places:

$$\boxed{\pm} \quad \boxed{a_0 \bullet a_{-1} a_{-2} \cdots a_{-k}}$$

where a_0 and $0 \le a_{-i} \le 9$ are natural numbers. I have elsewhere referred to such continua as heuristic pocket calculators [1, 2]. Their structure will be described more fully in Section 1.3.

The following hosts are unfamiliar and lie outside the scope of this paper.

(8) *Ring spectra* [21]. The numerical content of these host continua is their homotopy groups, but deeper structure is given by their geometry. (Algebraic topology is a branch of analysis, and the work of Eilenberg and MacLane was instrumental in laying its foundations.)

(9) *Line types* [12, 14]. These host continua, which do not exist in the category of sets, and whose theory is as yet incomplete, have sufficiently many elements d with $d^2 = 0$ to possess intrinsic differentiation of all functions. The tradition of discarding squares and higher powers of differentials is ancient in the calculus. It is manifested in the finite continua by the fact that $(10^{-k})^2 = 0$ in $C(k)$, for the square is numerically too small to be represented by anything but zero. If the theory of line types were correct, it would go far towards proving that both aspects of analysis, differentiation and integration and the theory of functions, were algebraic in nature.

This concludes our list of host continua. One obstacle to the understanding of the relationship between algebra and analysis has been the fact that, whereas algebraic theories exist in boundless variety, limited only by our ability to construct axiomatic systems, analysis has always appeared to the majority of mathematicians to be unique. The purpose of enumerating host continua has been, in part, to demonstrate that this is perhaps not the case. Although no attempt will be made to propose one in this paper, one may even imagine a *category* of distinct forms of analysis interrelated by morphisms. Within this heuristic category, the *combinatorial analysis* based on the host continua $C(k)$ appears to be an initial object. Indeed, there are inclusions

$$(10) \qquad C(k) \longrightarrow R$$

into all of the familiar host continua (since the displays in $C(k)$ are rational numbers $m/2^n5^p$, which exist in all host continua), and these inclusions are compatible with differentiation and integration. In combinatorial analysis, moreover, the fundamental theorem of calculus is valid without reference to infinite process of any kind. The construction, given in Section 2, relies wholly on finite simplicial sets and naive ideas borrowed from simplicial homotopy theory and numerical integration. The inclusions (10) can be continuously extended into the universe of R to give a morphism

$$(11) \qquad \text{holim } C \longrightarrow R$$

which is exactly universal for differentiation and integration. We lack a sufficiently precise general characterization of host continua to permit a good proof of this theorem, but for the familiar hosts, at least, the fundamental theorem of calculus can be transferred from holim C into R along this morphism. The morphism itself is determined by whatever concept of convergence reigns in R.

It is clear that a theorem of this kind must hold. For no matter what foundations may be adduced for analysis in R, the proof of such a fact as

$$\int_a^b \frac{dy}{dx}\, dx = y(b) - y(a)$$

always depends on the simple-minded idea of summing differences between consecutive values of y, as one counts along the x-axis. The extension of this idea to infinite continua R by means of a homotopy projective limit construction has no more effect on the essentially combinatorial nature of this process than has the extension of algebra to infinitary processes on the essential nature of algebra.

In Section 3 we return to the relationship between algebra and analysis, and reach a conclusion which is opposite to the usual one. Since the fundamental processes of analysis originate in combinatorics, we must understand the relationship between simplicial structures and the algebraic structures which enter analysis through the theory of functions. In effect, we have to squeeze algebra into the very small world of combinatorics. Unfortunately, free algebras won't fit. This destroys the equational structure of algebra. But there is still some useful structure lying in the ruins. The laws of algebra remain valid in a *coherence-theoretical* sense [18]. For us, this means valid up to coherent higher homotopies in the sense of [4]. Infinitely many homotopies must be simultaneously used to express this validity. We conclude in this slightly perverse way that it is really analysis which is finitary, and algebra that demands the recognition of actually infinite entities in mathematics.

1.3. *Some structures of finite continua*

$C(k)$ is a simplicial set. Its p-simplices are $(p+1)$-tuples $(x_0, x_1, \ldots, x_p)$ where the x_i are displays in $C(k)$. The simplicial operators are given by

$$\begin{aligned} \partial_i(x_0, x_1, \ldots, x_p) &= (x_0, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_p), \\ s_j(x_0, x_1, \ldots, x_p) &= (x_0, x_1, \ldots, x_j, x_j, \ldots, x_p), \end{aligned} \qquad p \geq 0.$$

For example, in geometric models the oriented difference between two displays x_0 and x_1 is represented by the 1-simplex (x_0, x_1) which has $\partial_0(x_0, x_1) = x_1$, $\partial_1(x_0, x_1) = x_0$. Thus $C(k)$ is convex.

The maps $C(k) \to R$ defined in the previous section will be considered as maps in the category of simplicial sets, where R also has the above simplicial structure (now $x_i \in R$).

The algebraic structure of $C(k)$ is defective. $C(k)$ cannot be a ring because the theory of rings is not combinatorial. But addition works well except for overflow; we always assume we remain within the numerical capacity of the continuum. Multiplication is neither distributive over addition nor associative. But $C(k)$ is a ring up to homotopy. Its deviations from the laws of algebra are decribed not as truncation errors but as homotopies in the above simplicial structure. It is of no relevance to combinatorial models whether these homotopies are short or not. The main concern is the coherence relations which they embody.

The maps $C(k) \to R$ are not compatible with the algebraic structure of $C(k)$, such as it is, for R is a simplicial ring.

Multiplication by powers of 10, however, shifts decimal points and does distribute over addition. It has the effect of mapping $C(k) \to C(k+l)$ where l is the power of 10. The value of k can also be changed by two simplicial maps

$$C(k) \underset{t}{\overset{i}{\rightleftarrows}} C(2k).$$

The inclusion i appends a block of k zeros and the truncation t deletes the final block of k digits. Intuitively, i is left adjoint to t, which contributes some coherent simplices to the structures. Some related maps will be denoted by the same letters in Section 2.3. It is convenient to double the precision so that the square of the step size at any level of precision is equal to the step size at the next.

Note that we did not take the trouble to bound the natural number a_0 in the $C(k)$ displays $\pm\ a_0 \cdot a_{-1} a_{-2} \cdots a_{-k}$ and the theory will not be strictly combinatorial until that is done. Much worse troubles arise with the simplicial set $BC(k)$ of Section 2, whose simplicial operators even seem to be in jeopardy. But it is easy to keep track of the bounds required. For ease of exposition we do not do this, and anyway, a better solution is indicated in Section 3.

The $C(k)$ play the role of host continuum. But no one $C(k)$ suffices. In fact, in combinatorial analysis there really is no host continuum. There is just a host *family*. Each member of the family is finite, but the family is infinite. Thus the apparent hapless dependence of analysis in $C(k)$ upon numerical manipulation is an illusion which actually leads our thinking nearer to the ideal of dealing with geometrical or physical quantity itself.

2. Combinatorial differentiation and integration

We only consider differentiation of functions and integration of differential forms on the unit interval. We shall imagine that we are trying to solve a differential equation $\mathrm{d}y/\mathrm{d}x = f(x)$ or $\mathrm{d}y = f(x)\mathrm{d}x$ subject to the initial condition $y(0)=0$, where $f(x)$ is a differential coefficient given numerically. The objects in terms of which the solution is expressed are related to the representing objects for functions and differential forms which arise in piecewise linear de Rham theory [6, 26, 27]. They are simultaneously examples of the "process" fibrations of [2] and heuristic hard-wired single-purpose processors which execute parts of the programming languages of [11, 23].

2.1. Functions, forms and differential coefficients

Let us fix a convenient number of decimal places, k, for the representation of the differential equation.

A *function* is any simplicial map $y : [0,1](k) \to C(2k)$ such that $y(0)=0$. We always differentiate and integrate rightwards, so the only p-simplices included in $[0,1](k)$ will be $(p+1)$-tuples

$$(x_0, x_1, \ldots, x_p) \quad \text{where } 0 \le x_0 \le x_1 \le \cdots \le x_p \le 1.$$

Since $C(2k)$ is convex, a function y is determined by its values on vertices:

$$(x_0, x_1, \ldots, x_p) \xrightarrow{y} (y(x_0), y(x_1), \ldots, y(x_p)).$$

The values $y(x_i)$ do not need to be in increasing order. Functions are always assumed to have double-precision values.

A *differential form* is a simplicial map $A : [0,1](k) \to BC(2k)$ where, in general, $BC(k)$ denotes the *classifying space* of $C(k)$ (see for example, [25]). A p-simplex of $BC(k)$ is any p-tuple $(x_1, x_2, \ldots, x_p)$ of displays of $C(k)$, and the simplicial operators are given by the formulas

$$\partial_i(x_1, x_2, \ldots, x_p) = \begin{cases} (x_2, \ldots, x_p), & i=0, \\ (x_1, \ldots, x_i + x_{i+1}, \ldots, x_p), & 1 \le i \le p-1, \\ (x_1, \ldots, x_{p-1}) & i=p, \end{cases}$$

$$s_j(x_1, x_2, \ldots, x_p) = (x_1, \ldots, x_j, 0, x_{j+1}, \ldots, x_p),\ 0 \le j \le p.$$

The *difference* map

$$C(k) \xrightarrow{\mathrm{d}} BC(k)$$

is defined by $\mathrm{d}(x_0, x_1, \ldots, x_p) = (x_1 - x_0, x_2 - x_1, \ldots, x_p - x_{p-1})$. The difference is a simplicial covering map (except for not being surjective because of the limited numerical range) and therefore has the property of unique lifting of homotopies (for all the homotopies we need).

We define the *differential* dy of a function y and the *integral* $\int A$ (from zero to x) of a differential form A by composition with d and lifting:

$$\begin{tikzcd} & C(2k) \arrow[d, "\mathrm{d}"] \\ {[0,1](k)} \arrow[ur, "y"] \arrow[r, "\mathrm{d}y"'] & BC(2k) \end{tikzcd} \qquad \begin{tikzcd} & C(2k) \arrow[d, "\mathrm{d}"] \\ {[0,1](k)} \arrow[ur, "\int_0^x A"] \arrow[r, "A"'] & BC(2k) \end{tikzcd}$$

where the integral lifting is uniquely determined by the condition $\int_0^0 A = 0$. In this context the fundamental theorem of calculus states that there is an isomorphism of simplicial sets

$$(\text{Differential forms}) \underset{\mathrm{d}}{\overset{\int_0^x}{\rightleftarrows}} (\text{Functions}).$$

A *differential coefficient* is a simplicial map $f: [0,1)(k) \to C(k)$. The half-open interval symbol indicates that no value $f(1)$ is required. A differential coefficient induces a differential form

$$[0,1](k) \xrightarrow{f(x)\mathrm{d}x} BC(2k)$$

The closed interval $[0,1](k)$ is (isomorphic to) a 10^k-simplex. A simplicial map on $[0,1](k)$ can be defined by prescribing its value arbitrarily on the highest-dimensional simplex (since the simplex is a free combinatorial object). We simply list the values of f multiplied by the stepsize:

$$\delta_k = (0, 10^{-k}, 2\cdot 10^{-k}, \ldots, (10^k - 1)10^{-k}, 1)$$
$$(f(x)\mathrm{d}x)(\delta_k) = (f(0)10^{-k}, f(10^{-k})10^{-k}, \ldots, f((10^k - 1)10^{-k})10^{-k})$$

$BC(2k)$ is not convex, so this map cannot be defined by specifying it on vertices.

Conversely, if A is a differential form, we get a differential coefficient $A/\mathrm{d}x$ by multiplying the 10^k entries of $A(\delta_k)$ by 10^k. Clearly $(A/\mathrm{d}x)\mathrm{d}x = A$ and $f(x)\mathrm{d}x/\mathrm{d}x = f(x)$. In fact, we have a further isomorphism of simplicial sets:

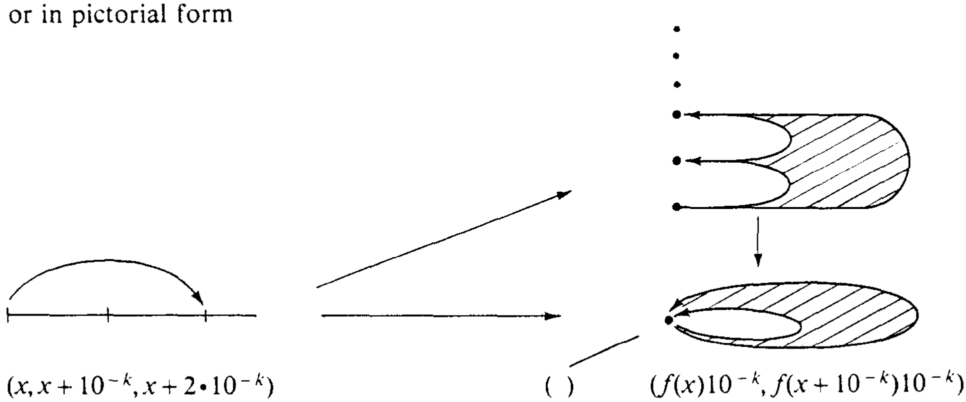
$$\begin{pmatrix}\text{Differential} \\ \text{Coefficients}\end{pmatrix} \underset{(\ )/\mathrm{d}x}{\overset{(\ )\mathrm{d}x}{\rightleftarrows}} \begin{pmatrix}\text{Differential} \\ \text{Forms}\end{pmatrix} \underset{\mathrm{d}}{\overset{\int_0^x}{\rightleftarrows}} (\text{Functions})$$

Using these isomorphisms we can successively transform the equation $\mathrm{d}y/\mathrm{d}x = f(x)$ into $\mathrm{d}y = f(x)\mathrm{d}x$ into $y = \int_0^x \mathrm{d}y = \int_0^x f(x)\mathrm{d}x$. (We won't bother to substitute a "dummy variable" for x under the integral sign.)

In diagrammatic form the solution is the lifting

$$\begin{tikzcd} & C(2k) \arrow[d, "\mathrm{d}"] \\ {[0,1](k)} \arrow[ur, "y = \int_0^x f(x)\mathrm{d}x"] \arrow[r, "f(x)\mathrm{d}x"'] & BC(2k) \end{tikzcd}$$

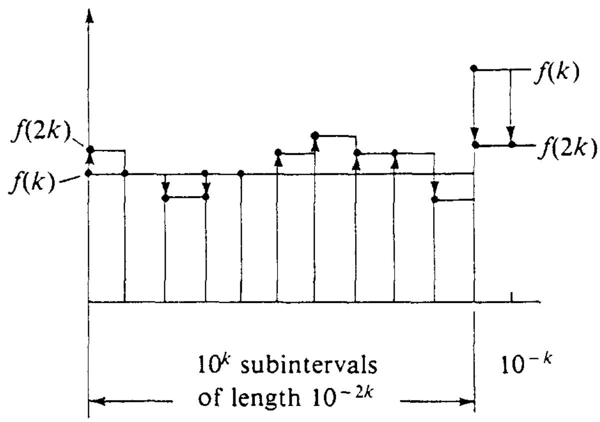
or in pictorial form



The simplicial set $BC(2k)$, viewed as a finite-state machine, has one state or vertex ( ), 1-dimensional edges or *loops* which are transitions from this state into itself, and higher-dimensional simplices which represent composites of loops. The 10^k-simplex $(f(x)\mathrm{d}x)(\delta_k)$ is a composite loop labeled with weighted values of the differential coefficient. When the path $f(x)\mathrm{d}x$ is lifted into $C(2k)$, the geometry forces the lifting of successive simplices in $[0,1](k)$ to correspond to algebraic addition of these weighted values in the fiber of d over ( ). This fiber is easily seen to be the set of values, i.e., the 0-skeleton, of $C(k)$. The classifying space for differential forms thus turns out to be a familiar tool of computation, the loop. (But the loop structure ought to be more explicitly analyzed in terms of the generating loop (10^{-k}) itself.)

2.2. Coherence

The preceding theory is limited to a single level of precision k in $[0, 1]$. How do the objects and morphisms change if the precision is doubled? We imagine that we have the "same" differential equation $\mathrm{d}y/\mathrm{d}x = f(x)$, but now $f(x)$ is measured to two levels of precision $f(k) : [0,1)(k) \to C(k)$ and $f(2k) : [0,1)(2k) \to C(2k)$. Let us draw a graph:



On macroscopic intervals where $f(k)$ is perceived as constant, $f(2k)$ has microscopic fluctuations. For each value of x in $[0,1)(2k)$ there is a numerical comparison of the earlier value $f(tx;k)$ and the new value $f(x;2k)$. Since $C(k)$ is convex, these numerical comparisons give a homotopy in the square

$$\begin{tikzcd}
{[0,1)(k)} \arrow[r, "f(k)"] & C(k) \arrow[d, "i"] \\
{[0,1)(2k)} \arrow[u, "t"] \arrow[r, "f(2k)"'] & C(2k)
\end{tikzcd} \tag{1}$$

that is, a simplicial map $\Delta(1)\times[0,1)(2k)\to C(2k)$.

In the same way, if we have a sequence of more precise measurements $f=(f(2^jk))_{0\le j\le m}$, then these define a ladder of homotopy-commutative squares. These squares compose to give an m-dimensional homotopy denoted

$$\Delta(m)\times[0,1)(2^mk)\xrightarrow{(f)} C(2^mk)$$

which is a higher homotopy of the maps

$$\begin{tikzcd}
{[0,1)(2^jk)} \arrow[r, "f(2^jk)"] & C(2^jk) \arrow[d, "i"] \\
{[0,1)(2^mk)} \arrow[u, "t"] \arrow[r, "f(2^mk)"'] & C(2^mk)
\end{tikzcd}$$

At the same time (f) is an m-simplex in the simplicial set of differential coefficients of precision 2^mk. We therefore get an m-simplex of differential forms

$$\Delta(m)\times[0,1](2^mk)\xrightarrow{(f(x)\mathrm{d}x)} BC(2^{m+1}k)$$

which is a higher homotopy of the maps

$$\begin{tikzcd}
{[0,1](2^jk)} \arrow[r, "f(2^jk)\mathrm{d}x"] & BC(2^{j+1}k) \arrow[d, "i"] \\
{[0,1](2^mk)} \arrow[u, "t"] \arrow[r, "f(2^mk)\mathrm{d}x"'] & BC(2^{m+1}k)
\end{tikzcd}$$

Finally, by integration (lifting), we get an m-simplex of functions

$$\Delta(m)\times[0,1](2^mk)\xrightarrow{(y)} C(2^{m+1}k)$$

which is a higher homotopy of functions obtained by integration at lower levels:

$$\begin{tikzcd}
{[0,1](2^jk)} \arrow[r, "y(2^jk)"] & C(2^{j+1}k) \arrow[d, "i"] \\
{[0,1](2^mk)} \arrow[u, "t"] \arrow[r, "y(2^{m+1}k)"'] & C(2^{m+1}k)
\end{tikzcd}$$

All of the last three squares exist for $0 \leq j \leq m$, and these are the coherence relations we seek. There is no way to build an obstruction; all of the complexes which parametrize the coherence are, and in this paper will remain, contractible. (They are the above simplices.)

Homotopy coherence, however, does give rise to a modest host continuum in which the coherent differentiation and integration processes take values. This host is not laid down *a priori* but evolves naturally from these processes themselves. It comes from exponential adjointness and the idea of homotopy inverse limits [4, 7, 28].

Consider the directed system

$$C(2k) \xleftarrow{t} C(4k) \xleftarrow{t} \cdots \xleftarrow{t} C(2^{j+1}k) \xleftarrow{t} \cdots \xleftarrow{t} C(2^m k)$$

The ordinary inverse limit is built from sequences of elements $(y(2^{j+1}k))$ which fit together under truncation. Such an element is nothing but a $2^m k$-place decimal divided into blocks of digits which are to be retained/discarded.

In the *homotopy* inverse limit, by contrast, the elements of the sequences only have to fit together up to coherent homotopy. This corresponds to the fact that in our case the homotopies

$$\begin{tikzcd}
{[0,1](2^j k)} \arrow[r, "y(2^j k)"] & C(2^{j+1}k) \arrow[d, "i"] \\
{[0,1](2^{j+1}k)} \arrow[u, "t"] \arrow[r, "y(2^{j+1}k)"'] & C(2^{j+2}k)
\end{tikzcd}
\quad \text{or} \quad
\begin{tikzcd}
{[0,1](2^j k)} \arrow[r] & C(2^{j-1}k) \\
{[0,1](2^{j+1}k)} \arrow[u, "t"] \arrow[r] & C(2^{j-2}k) \arrow[u, "t"']
\end{tikzcd}
\tag{2}$$

(same horizontals)

can be jointly extended to the m-simplex. Technically, this means that in the following diagram there exist (unique) factorizations denoted y, $f(x)\mathrm{d}x$:

$$\begin{tikzcd}
& \text{holim } C \arrow[rr] \arrow[d, "\mathrm{d}"] & & C(2^{m-1}k)^{\Delta(m)} \arrow[d, "\mathrm{d}"] \\
{[0,1](2^m k)} \arrow[ur, "y"] \arrow[r, "f(x)\mathrm{d}x"'] \arrow[urrr, bend left, "\text{adjoint }(y)"] \arrow[rrr, bend right, "\text{adjoint }(f(x)\mathrm{d}x)"'] & B(\text{holim } C) \arrow[r] & B(C(2^{m+1}k)^{\Delta(m)}) \arrow[r] & (BC(2^{m-1}k))^{\Delta(m)}
\end{tikzcd}$$

The outside maps are the exponential adjoints of the higher homotopies obtained before, and the symbol holim C denotes the homotopy inverse limit of the sequence $C(2^{j+1}k)$ under truncation $(0 \leq j \leq m)$.

Thus we see that coherent differential coefficient data $f = (f(2^j k))$ can be integrated to coherent function data $y = (y(2^{j+1}k))$, and this function data is the solution of the differential equation $dy/dx = f(x)$ in the sense of a commutative diagram like that obtained before:

$$
\begin{tikzcd}
 & \operatorname{holim} C \arrow[d, "d"] \\
{[0,1](2^m k)} \arrow[ur, "y=\int_0^x f(x)dx"] \arrow[r, "f(x)dx"'] & B(\operatorname{holim} C)
\end{tikzcd}
$$

Nothing in this pattern will change if the parameter m is allowed to tend to infinity.

The values of the "function" y cannot be interpreted as real numbers in any ordinary sense. This is due to the wild, in fact quite arbitrary, fluctuations of their truncations which homotopy coherence permits. The following is an illustration of an element in holim C when $k = 1$ and $m = 2$:

$$
\begin{tikzcd}
3.14 & 3.1415 \arrow[l] & \mathit{3.14159265} \arrow[l] \\
2.71 \arrow[u] & \mathit{2.7182} \arrow[l] \arrow[u] & \\
\mathit{0.57} \arrow[u] & &
\end{tikzcd}
$$

These elements have only a formal use which is like that of the infinite strings of integers which arise in [8, sections 3–5]. We prefer to work with homotopy-theoretic coherence rather than with the algebraic coherence obtained by dividing by a carrying ideal, since the theory of rings is not combinatorial. This also prevents infinite collapsing of strings as occurs in [8, p. 283].

2.3. *Relation with differentiation and integration in other continua*

Suppose that we have a differential equation $dy/dx = f(x)$ $(y(0) = 0)$ where $f : [0,1](R) \to R$ and R is one of the familiar continua. We also write y_R, f_R to distinguish y, f from their combinatorial counterparts, for which we retain the notation of previous sections. We want to construct a diagram which relates the combinatorial solution of the equation with the solution in R. The same idea is essentially valid in all cases, but for brevity we only treat $R =$ (classical real numbers).

When the fineness of space parameter m of Section 2.2 tends to infinity, the maps $[0,1](2^m k) \to [0,1](R)$ subdivide the classical unit interval more and more finely and ultimately induce a map of the (ordinary) inverse limit

$$
\begin{aligned}
[0,1](\infty) = \lim_m\ [0,1](2^m k) &\xrightarrow{\ i\ } R \\
= [0,1](\text{infinite decimals})&
\end{aligned}
$$

Similarly, when $m \to \infty$, we obtain the infinite homotopy limit as the (ordinary) inverse limit of the finite homotopy limits constructed in (2.2); we now write

$$\operatorname{holim} C = \lim_{m} \operatorname{holim} C(2^{j+1}k) \quad (0 \leq j \leq m)$$

where the limit is taken over the maps

$$\operatorname{holim}(0 \leq j \leq m) \xleftarrow{t} \operatorname{holim}(0 \leq j \leq m+1)$$

which forget the last component of a homotopy coherent string $(y(2^{j+1}k))$ $(0 \leq j \leq m+1)$. All of our constructions pass to the limit over this system. It is clear that holim C now just consists of infinite homotopy coherent strings. If we restrict to those strings with the property that $\lim y(2^{j+1}k)$ exists in R, then we get a map from what we shall call the *convergent* subobject of holim C:

$$\operatorname{holim} C \longleftarrow (\operatorname{holim} C)_{\text{conv}} \xrightarrow{i} R$$

Note that the objects $[0,1](2^m k)$ and $\operatorname{holim} C(2^{j+1}k)$ $(0 \leq j \leq m+1)$ embody the idea of potential infinity, but the inverse limit objects are the first actually infinite mathematical entities to appear, and they do so in the context of making the comparison with the actually infinite continuum R.

We now construct the following diagram, omitting many details.

$$\begin{tikzcd}
 & (\operatorname{holim} C)_{conv} \arrow[dd, "i"] \arrow[dr, "d"] & \\
{[0,1](\infty)} \arrow[ur, "y"] \arrow[rr, "f(x)\mathrm{d}x"'] \arrow[dd, "i"'] & & B(\operatorname{holim} C)_{conv} \arrow[dd, "Bi"] \\
 & R \arrow[dr, "d"] & \\
{[0,1](R)} \arrow[ur, "y_R"] \arrow[rr, "f_R(x)\mathrm{d}x"'] & & BR
\end{tikzcd}$$

(2)

(3)

Triangle (2) is obtained as follows. The differential coefficient f_R induces combinatorial differential coefficients by decimal inclusion and truncation:

$$\begin{tikzcd}
{[0,1)(2^j k)} \arrow[r, "f(2^j k)"] \arrow[d, "i"'] & C(2^j k) \\
{[0,1](R)} \arrow[r, "f_R"'] & R \arrow[u, "t"']
\end{tikzcd}$$

If $r \in R$, $t(r)$ is the greatest display of $C(2^j k)$ which is $\leq r$; t induces a map of the simplicial structure imposed on R in Section 1.3. On the level of points, t looks discontinuous. (It is assumed that some appropriate level of precision k is fixed at the outset; our notation, descended from Section 1.2, allows some latitude in this choice.)

The infinite string $(f(2^j k))$ is coherent up to homotopy. There results an inverse system of commutative integration diagrams

$$\begin{tikzcd} & \operatorname{holim} C(2^{j+1}k) \arrow[dr, "d"] & \\ 0,1(2^m k) \arrow[ur, "y"] \arrow[rr, "f(x)\mathrm{d}x"'] & & B \operatorname{holim} C(2^{j+1}k) \end{tikzcd} \quad (0 \le j \le m)$$

connected by the truncation maps (1). Triangle (2) is the inverse limit of this system and is therefore also commutative.

Conditions on f_R must be imposed in order to ensure that its infinite combinatorial integral y in (2) factor through the convergent subobject. This is standard when f_R is a classical continuous function. One uses uniform continuity on $[0,1](R)$ and estimates of truncation error to show that the homotopies in diagrams (1) and (2) (Section 2.2) are "less than ε", thus that the values $(y(x;2^{j+1}k))$ lie in the convergent subobject. (In this case we really *are* talking about truncation error, because we have the classical reals to refer to.)

The case $R =$ (infinite decimals) is too complicated to go into. It is curious that no intrinsic definition of continuity of f_R has ever been given, however, nor theory of the resulting integral. (By "intrinsic" I mean based on the infinite branching structure which the numbers in this continnum have.) In this case it is probably natural to *assume* that f_R is built up from finite combinatorial differential coefficients like those we have considered. In particular, if f_R is built up by a self-replicating (or "fractal" [19]) process, then the solution function y has the same character, and factors through an appropriately defined convergent subobject. Note that the numbers in $(\operatorname{holim} C)_{\mathrm{conv}}$ defined above need not stabilize decimally because of the usual problem of nines. The following element

$$\text{“}0.99 \longleftarrow 1.0000 \longleftarrow 0.99999999 \longleftarrow \ldots\text{”}$$

converges to $1 \in R$ (classical). However, the fibers of the maps

$$[0,1](\infty) \xrightarrow{i} [0,1](R) \quad \text{and} \quad \operatorname{holim} C \xrightarrow{i} R$$

are contractible.

We forgot to mention that if f_R is a classical *Lipschitz* function, then the values of y converge at a rapid rate.

To define the maps in triangle (3), choose a cross section t of $[0,1](\infty) \to [0,1](R)$. For example, we can take for $t(r)$ the greater decimal representative of r. (In fact, $[0,1](R)$ is a simplicial deformation retract of $[0,1](\infty)$.) Let $y_R = i \circ y \circ t$ and $f_R(x)\mathrm{d}x = i \circ f(x)\mathrm{d}x \circ t$. By restriction to the 0-skeleton, y_R can be interpreted as a set-theoretic mapping, and it is easy to check that its derivative in the classical sense, y_R', coincides with the originally given continuous differential coefficient f_R (by continuity of the latter).

In combinatorial analysis the concepts of cochain and differential form coincide.

Since we insisted on representability of both concepts, our procedure leads to the nearest *representable* classical analogues. The simplicial set BR is an Eilenberg–MacLane object $K(R, 1)$, and the simplicial map $f_R(x)\mathrm{d}x$ is the 1-cochain on $[0, 1](R)$ which results from the (de Rham) integration map of differential forms into real cochains.

2.4. The differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \quad (y(0) = 0). \tag{1}$$

There are two well-known classical methods for solving this equation.

Method of successive approximations. If $f(x, y)$ satisfies a Lipschitz condition in y, then the integral operator

$$(Ty)(x) = \int_0^x f(x, y(x))\mathrm{d}x \tag{2}$$

is a contraction on a complete metric space of functions. The sequence of successive approximations $y_{(0)} = 0$, $y_{(n+1)} = Ty_{(n)}$ therefore converges to a unique fixed point y of T. Thus y has the property

$$y(x) = \int_0^x f(x, y(x))\mathrm{d}x \tag{3}$$

which is equivalent to (1). The function y is the (unique) solution of (1).

It is just bad luck that the function $y(x)$ appears on both sides of (3), so that (3) is only equivalent to (1), and does not give a way of finding the solution. Using induction over the external natural number object is ingenious, but it is hard to see why the classical structure of space should be so weak as to make thus necessary. Note that in a concrete realization of this method the passage $n \to \infty$ would be controlled by an infinite loop on n.

Euler's method. Divide the interval $[0, 1](R)$ by 0-simplices $x_0 = 0, x_1, x_2, \ldots, x_M = 1$. Then the values

$$\bar{y}(x_{i+1}) = \sum_{i=0}^{M} f(x_i, \bar{y}(x_i))(x_{i+1} - x_i), \tag{4}$$

linearly interpolated, provide a piecewise linear approximation to the solution, $\bar{y}$. When $M \to \infty$ and $\max(x_{i+1} - x_i) \to 0$, the Lipschitz condition implies that $\bar{y}$ tends to a differentiable function which is the desired solution, y.

Euler's method is easy to mimic in finite continua. But since these are already canonically subdivided, equations (3) and (4) coincide. The integral in (3) is evaluated by the B-complex looping process described in Section 2.1. Although $y(x)$ appears on both sides of the equation, the integral evaluation only makes use of previously obtained values. (A similar idea is used in the non-standard analysis treatment of this problem [24, p. 267 ff.].) Note that $y(x)$ is also an exact fixed point for the combinatorial integral operator T.

Let us model the method of successive approximations in the system $C(2^{j+1}k)$ $(0 \leq j \leq m)$. The functions and the integral operator are interpreted combinatorially. It is easy to see that the iteration on n stabilizes after (an easily computed exponential function of m) steps. In order to approximate closely to the classical solution, we must let $m \to \infty$ and this would also be controlled by an infinite loop, now on m. (The fineness of space parameter m corresponds to the length of the summation in (4).) Stabilization of the iteration means that the m-loop and the n-loop are *nested*. Thus, in reality, there is only one significant parameter, m, and the external induction on n has been absorbed into the structure of combinatorial space.

The sufficiency of the Lipschitz condition is both classically and combinatorially clear. However, the existence and uniqueness of solutions can classically be proved under weaker, log-Lipschitz, conditions. [10, p. 67] gives

$$|f(x,y_1) - f(x,y_0)| < K|y_1 - y_0| \log \left| \frac{1}{y_1 - y_0} \right|$$

and others with more iterated logarithms. When the difference $y_1 - y_0$ is replaced with a canonical stepsize, these correspond to the "fractal" convergence mentioned at the end of Section 2.3.

3. The relation with algebra, again

To understand the relationship between algebra and analysis we should investigate the interaction of combinatorial theories, with their finite free object constructions, and algebraic theories, with their generally infinite ones. We recall that the standard categorical constructions of free algebras, coequalizers, resolutions and left adjoints to algebraic functors all make explicit use of an actually infinite mathematical object (the object of natural numbers). On the other hand, if differentiation and integration are carried out geometrically, as Barrow knew how to do, or if the combinatorial interpretations of Section 2 are accepted, it appears that the fundamental processes of analysis do not.

The difference between the intrinsic infiniteness of algebra and the intrinsic finiteness of analysis becomes particularly striking when we consider the role which algebra plays *within* analysis. Not only are there formulas in calculus which relate algebraic operations on functions to differentiation and integration, such as

$$\frac{d}{dx}(yz) = \frac{dy}{dx}z + y\frac{dz}{dx}, \tag{1}$$

there are even others in which algebraic operations seem indispensable for statements which ought to be purely analytical, for example

$$\frac{dz}{dx} = \left(\frac{dz}{dy}\right)\left(\frac{dy}{dx}\right). \tag{2}$$

The presence of such compatibility relations is an important aspect of the relationship between algebra and analysis.

If a relation between two mathematical objects is sought, the strongest relation will be obtained by ascent from the deepest conceptual level at which both objects exist. Thus, if differentiation and integration are combinatorial processes, we have to remodel algebraic processes combinatorially in order to obtain the compatibility relations we seek. One method is to build combinatorial models of algebraic processes as fibrations over simplicial nerves of algebraic theories. (See [2] for some indications.) Composition of operations, indefinite in algebra, corresponds to lifting of successive simplices over this nerve, but the latter is always a finite process. The finiteness of the combinatorial natural number object (the simplex, the element of space) forces compatibility statements like (1), (2), and even purely algebraic laws, to emerge in the form of category-theoretical (or homotopy-theoretical) coherence diagrams, rather than equations.

Difficulties of this kind seriously affect the calculator objects $C(k)$ and their classifying complexes $BC(k)$. It is not really right to borrow the B-complex from the theory of simplicial abelian groups or categories, because on account of overflow $C(k)$ is not an abelian group, nor can sums of arbitrary length be formed. Some of the face operators in $B(k)$ are therefore not well-defined, and there are infinitely many simplices which never play any role in a given integration. But if the addition operation were modelled combinatorially (which would involve more loops), it would be easy to include specific structures which would enlarge the system automatically, or better, build the host objects step by step as needed. It seemed unenlightening to enter into such complications, which, however, are not unreal.

To relate algebraic operations to analysis, we must verify coherence, or contractibility, of many complexes which keep track of operational complexity. It is ordinarily necessary to build actually infinite dimensional complexes in order to kill off all obstructions. But plenty of homotopies are available in the convex finite continua. The laws of algebra, then, and the correct relations with differentiation and integration will hold if these complexes can be contracted to points without offense to the topological forms of the structures. It is precisely the point of categorical coherence theory to prove that this is possible.

In other circumstances in analysis such contractions do not exist. In these cases it is precisely the point of categorical coherence theory to determine the automorphisms of objects induced by coherence, and the non-trivial homotopy-theoretical obstructions which result. The coherence theory of an isomorphism-commutative bifunctor, which gives rise to stable homotopy theory, is a good example of this. The existence of classical continuous functions which are not classically differentiable is another. Thus categorical coherence theory, originated by Mac Lane, is of far greater relevance to the foundations of analysis than may be supposed.

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